§ 4 Theories of Failure

30 Introduction. If a ductile metal bar is subjected to a gradually increasing axial tensile load causing only one principal stress on any transverse section, the material, when the load reaches a certain value, will begin to acquire inelastic (permanent) deformation.

In Art. 4 it was assumed that inelastic action after it had progressed to a small (measurable) amount constituted structural damage to the member and was designated as failure by general yielding; it was attributed primarily to slip on planes through the crystalline grains of

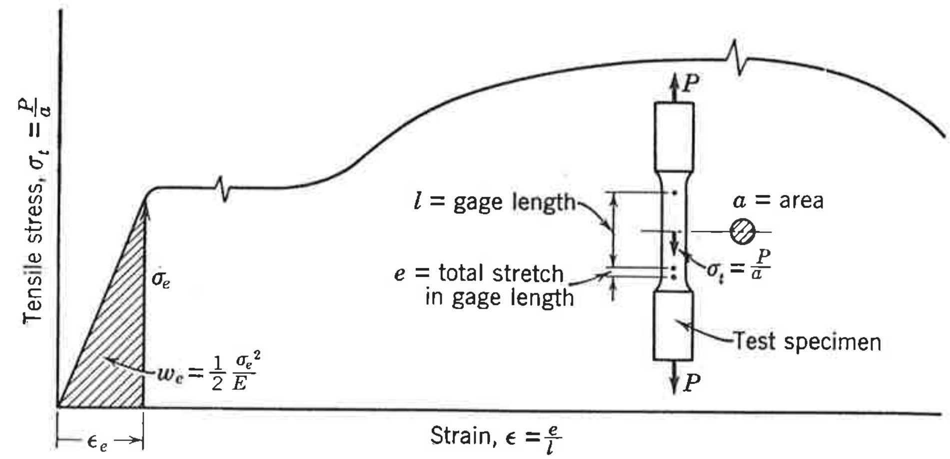


FIG. 44 Typical stress-strain diagram for ductile steel.

the metal. It was assumed also that the slip (yielding) was more closely associated with shearing stress than with any other quantity, and hence a limiting value of the shearing stress (the shearing elastic limit or yield point) was considered to be the property of the material which would limit the load-resisting capacity of a member made of ductile material. There are, however, at least five other quantities or properties of the material that have been proposed and used in design as a measure of the limiting resistance value or maximum utilizable strength of the material when the beginning of yielding is the action that destroys the load-resisting function of the member. Some of these theories are used also to explain failure by fracture, as will be discussed later.

In Fig. 44 is shown a typical tensile stress-strain curve for a specimen of ductile steel as obtained from a tension test. When the specimen starts to yield, the following six quantities are reached simultaneously:

1. The principal stress ($\sigma = P/a$) reaches the tensile elastic strength (elastic limit or yield point) σ_e of the material.

2. The maximum shearing stress [$\tau = \frac{1}{2}(P/a)$] reaches the shearing elastic limit or yield point τ_e of the material, $\tau_e = \frac{1}{2}\sigma_e$.

3. The tensile strain ϵ reaches the value ϵ_e.

4. The total strain energy w absorbed by the material per unit volume reaches the value $w_e = \frac{1}{2}(\sigma_e^2/E)$.

5. The strain energy of distortion w_d (energy accompanying change in shape) absorbed by the material per unit volume reaches a value $w_{de} = [(1 + \mu)/3E]\sigma_e^2$.

6. The octahedral shearing stress reaches the value $\tau_{Ge} = (\sqrt{2}/3)\sigma_e = 0.47\sigma_e$.

These six criteria of failure of a material are summarized in Table 1.

The six limiting values given in Table 1 occur simultaneously in a tensile specimen, in which the state of stress is uniaxial, and hence it is impossible to determine from a tension test which one of the quantities is the cause of the beginning of inelastic action. If, however, the state of stress is biaxial or triaxial, the foregoing six quantities will not occur simultaneously, and it is a matter of considerable importance in design as to which one of the quantities is assumed to limit the loads that can be applied to a member without causing inelastic action. The six limiting quantities as given in Table 1 suggest six theories of failure or six different methods for using data obtained in the tension test to predict inelastic action when the state of stress in the member is not uniaxial. These theories of failure are discussed in the next article.

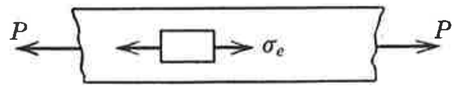
Before stating the theories of failure it may be well to recall (from Chapter 1) that failure of load-resisting members, under static loading as here considered, usually consists of one of two types of action, namely, (*a*) inelastic deformation (yielding) or (*b*) brittle fracture, by which is meant separation of the material without accompanying measurable yielding. Which one of these two modes of failure occurs depends on the inherent, *internal* characteristics and structure of the material, and also on *external* conditions, such as temperature, state of stress, type of loading, rate of loading, etc.; a stress-strain diagram for a material that fails by truly brittle fracture under static loading is a straight line until the breaking or fracture stress is reached. In such a failure the elastic limit and the ultimate strength of the material are identical values. The significance of any theory of failure will depend to a considerable extent on which mode of failure occurs or is assumed to occur.

31 Statement of Theories of Failure. The six main theories of failure suggested in Table 1 for a material that is considered to fail by yielding under static loading may be stated briefly as follows:

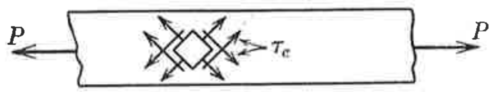
1. The *maximum principal stress* theory, often called Rankine's theory, states that inelastic action at any point in a material at which any state of stress exists begins *only* when the maximum principal stress at the

TABLE 1

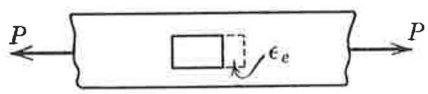
Theory of Failure	Maximum Utilizable Quantity as Obtained from Tension Test
1. Maximum Principal Stress	$\sigma_e = P/a$



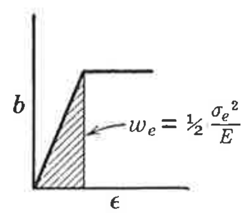
2. Maximum Shearing Stress	$\tau_e = \frac{1}{2}(P/a) = \frac{1}{2}\sigma_e$



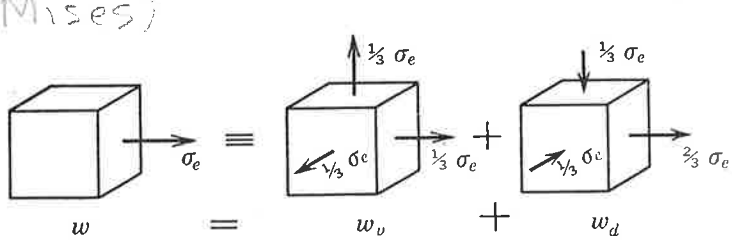
3. Maximum Strain	$\epsilon_e = \sigma_e/E$



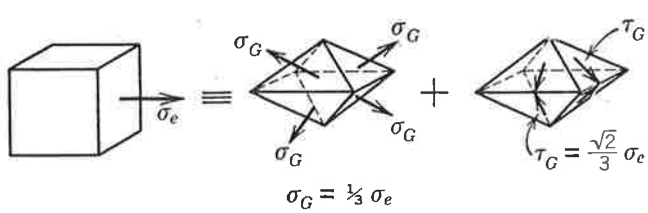
4. Total Strain Energy	$w_e = \frac{1}{2}(\sigma_e^2/E)$



5. Strain Energy of Distortion (von Mises)	$w_{de} = \frac{1+\mu}{3}\frac{\sigma_e^2}{E}$



6. Octahedral Shearing Stress	$\tau_{Ge} = (\sqrt{2}/3)\sigma_e = 0.47\sigma_e$



point reaches a value equal to the tensile (or compressive) elastic limit or yield strength of the material as found in a simple tension (or compression) test, regardless of the normal or shearing stresses that occur on other planes through the point. Thus, according to this theory, if the block in Fig. 46*a* reaches its elastic limit when subjected to the stress σ_1, the elastic limit will still be σ_1 even if the block is subjected to the stress σ_2 (Fig. 46*b*) in addition to σ_1.

It will be observed that if σ_1 and σ_2 are equal and of opposite sign, shearing stresses τ equal to σ will be developed on 45° diagonal planes

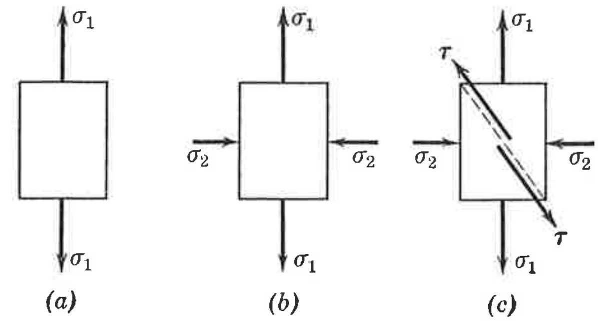


FIG. 46

as in Fig. 46*c*. A state of stress like that shown in Fig. 46*c* occurs in a cylindrical bar subjected to pure torsion. Thus, if this theory is true for all states of stress, the shearing elastic limit of the material must be at least equal to the tensile elastic limit. But for all ductile metals the shearing elastic limit as found from the torsion test is much less than the tensile elastic limit as found from the tension test. It is evident, therefore, that the presence of relatively large shearing stresses at a point causes limitations on the maximum principal stress theory which will be discussed further in the next article. For brittle materials which do not fail by yielding but fail by brittle fracture, the maximum principal stress theory is considered to be reasonably satisfactory, although the maximum strain theory is considered to be preferable.

2. The *maximum shearing stress* theory, sometimes called Coulomb's theory, or Guest's law, states that inelastic action at any point in a body at which any state of stress exists begins *only* when the maximum shearing stress on some plane through the point reaches a value equal to the maximum shearing stress in a tension specimen when yielding starts. This means that the shearing elastic limit must be not more than one-half the tensile elastic limit, since the maximum shearing stress in a tension specimen (on a 45° oblique plane) is one-half the maximum tensile stress in the specimen.

The maximum shearing stress theory seems to be fairly well justified for ductile material and for the states of stress encountered in most load-resisting members, that is, for states of stress in which relatively large shearing stresses are developed. However, for the state of stress of pure shear in which the maximum amount of shear is developed, as occurs in a torsion test, the shearing elastic limit of ductile metals is found to vary somewhat, but an average value is approximately 0.57 of the tensile elastic limit ($\tau_e = 0.57\sigma_e$), and hence for such a state of stress the maximum shearing stress theory errs (on the side of safety) by approximately 15 per cent.

In Art. 21 and Figs. 25*g*, 25*h*, and 25*i*, it is shown that maximum and minimum principal stresses may be resolved into a state of pure shear combined with equal tensions in all directions in the plane of these two principal stresses. Thus it is assumed by this theory of failure that the maximum shearing stresses alone produce inelastic action and that the equal tensile stresses have no influence in starting inelastic action. If the state of stress consists of triaxial tensile stresses of nearly equal magnitude, shearing stresses would be very small and failure would be by brittle fracture rather than by yielding, and hence the maximum shearing stress theory would not be applicable.

3. The *maximum strain* theory, often called St. Venant's theory, states that inelastic action at a point in a body at which any state of stress exists begins *only* when the maximum strain at the point reaches a value equal to that which occurs when inelastic action begins in the material under a uniaxial state of stress, as occurs in the specimen in the tension test. This value, ϵ_e, occurs simultaneously with the tensile elastic limit σ_e of the material. Thus $\epsilon_e = \sigma_e/E$.

For example, according to this theory, inelastic action in the block of Fig. 46*a* begins when σ_1 becomes equal to σ_e since $\epsilon_e = \sigma_e/E$, but in Fig. 41 $\epsilon = (\sigma_1/E) - \mu(\sigma_2/E)$, and hence inelastic action does not begin until σ_1 becomes greater than σ_e, since the strain in the direction of σ_1 is decreased by the amount $\mu(\sigma_2/E)$. Therefore, according to this theory of failure, σ_1 could be increased to a value somewhat higher than σ_e without causing yielding if the second normal stress σ_2 is a tensile stress, but if σ_2 is a compressive stress the maximum value of σ_1 that could be applied without causing yielding would be somewhat smaller than σ_e (Fig. 46*b*).

The maximum strain theory of the breakdown of elastic action is an improvement over the maximum principal stress theory, but, like the latter theory, it usually is not applicable if the failure in elastic behavior is by yielding; it is primarily applicable when the conditions are such that failure occurs by brittle fracture.

4. The *total energy* theory, proposed by Beltrami and by Haigh, states

that inelastic action at any point in a body due to any state of stress begins *only* when the energy per unit volume absorbed at the point is equal to the energy absorbed per unit volume by the material when subjected to the elastic limit under a uniaxial state of stress, as occurs in a simple tensile test. As shown by Eq. 51 and in Table 1 the value of this maximum energy per unit volume is $w_e = \frac{1}{2}(\sigma_e^2/E)$. The expressions for the total energy absorbed per unit volume for various states of stress are given in Art. 28; according to the total energy theory none of these expressions can exceed the value $\frac{1}{2}(\sigma_e^2/E)$ without causing yielding to start.

5. The *energy of distortion* theory, which grew out of the analytical work of Huber, von Mises, and Hencky and out of the results of tests by Bridgman on various materials showing that the material did *not* become inelastic under a triaxial state of stress produced by very high hydrostatic pressure, states that inelastic action at any point in a body under any combination of stresses begins *only* when the strain energy of distortion per unit volume absorbed at the point (see Art. 30) is equal to the strain energy of distortion absorbed per unit volume at any point in a bar stressed to the elastic limit under a state of uniaxial stress as occurs in a simple tension (or compression) test. As shown by Eq. 63 and in Table 1, the value of this maximum strain energy of distortion (energy absorbed in changing shape) as determined from the tension test is $w_{de} = [(1 + \mu)\sigma_e^2]/3E$.

The maximum energy of distortion theory differs from the maximum total energy theory as follows. In the maximum total energy theory it is assumed that the entire strain energy is associated with the beginning of inelastic action. However, tests of various materials under very high hydrostatic stresses show that the materials could withstand, without inelastic action taking place, strain energy values many times greater than those obtained in the simple axial load compression test. Hence, since in the hydrostatic tests the total strain energy is used in producing volume changes only, it was proposed that the energy absorbed in changing volume has no effect in causing failure by yielding, and that failure by inelastic action is associated only with energy absorbed in changing shape. It is assumed that if it were possible to make tests of materials under a negative hydrostatic pressure, which would create three *equal tensile* principal stresses, the same results as found for three equal principal compressive stresses would be obtained; that is, no yielding would take place, although fracture eventually would occur. Since change of shape involves shearing stresses, the energy of distortion theory is sometimes called (somewhat erroneously) the shear energy theory.

In a state of stress in which the maximum amount of shear exists, as in the state of pure shear shown in Fig. 46*c*, the energy of volume change is zero, because in Eq. 58 the average principal stress σ_{avg} is equal to zero, and therefore the total strain energy is used in distorting (changing the shape of) the unit volume. The expressions for the energy of distortion per unit volume for various states of stress are given in Art. 29, and, according to this theory, none of these expressions can exceed the value $[(1 + \mu)\sigma_e^2]/3E$ without causing the material to start to yield.

6. The *octahedral shearing stress theory* gives the same results as does the energy of distortion theory and hence may be called an *equivalent stress* theory. The octahedral shearing stress as given by Eq. 33 can be expressed in terms of the energy of distortion w_d. This is done by multiplying and dividing the right side of Eq. 33 by the quantity $\sqrt{(1 + \mu)/6E}$. Equation 33 then becomes

$$\tau_G = \frac{1}{3}\sqrt{\frac{6E}{1 + \mu}} \cdot \sqrt{\frac{1 + \mu}{6E}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]} \quad (67)$$

By referring to Eq. 60 it will be noted that Eq. 67 may be rewritten

$$\tau_G = \tfrac{1}{3}\sqrt{6Ew_d/(1 + \mu)} \quad (68)$$

where w_d is the energy of distortion. But the criterion of failure according to the maximum energy of distortion theory is that inelastic action begins when w_d becomes equal to $w_{de} = [(1 + \mu)\sigma_e^2]/3E$ (see Eq. 61). By substituting this value of w_{de} in Eq. 68, the octahedral shearing stress is found to be

$$\tau_G = (\sqrt{2}/3)\sigma_e = 0.47\sigma_e \quad (69)$$

It will be observed that the value of τ_G as required by the maximum energy of distortion theory in Eq. 69 is the same as the value given in Eq. 27 for the octahedral shearing stress that occurs in the standard tensile test. Thus an octahedral shearing stress theory may be stated as follows: Inelastic action at any point in a body under any combination of stresses begins *only* when the octahedral shearing stress τ_G becomes equal to $0.47\sigma_e$, where σ_e is the tensile elastic strength of the material as determined from the standard tension test. The octahedral shearing stress theory of failure makes it possible to apply the energy of distortion theory of failure by dealing only with stresses instead of dealing with energy directly; this procedure to some engineers seems desirable because stress is a more familiar quantity in engineering design than is energy.

Another way of interpreting the effect of the octahedral stresses has been given in Art. 24 and Figs. 34*d*, 34*e*, and 34*f*, where it was shown that any state of stress consisting of three principal stresses may be resolved into two component states of stress: one component consists of equal tensile (or compressive) stresses in all directions which does not influence the starting of inelastic action, but which may produce fracture, and the other component state of stress comprises the eight octahedral shearing stresses which are assumed by this theory to be wholly responsible for starting inelastic action.

32 Significance of the theories of failure. In Art. 2 it was pointed out that the rational procedure of design of a member requires that the general mode of failure of the member under the assumed service conditions be determined or assumed (failure usually is by yielding or by fracture) and that a quantity (stress, strain, or energy, etc.) be chosen which is considered to be associated with the failure. This means that there is a maximum or critical value of the quantity selected which limits the loads that can be applied to the member; furthermore, it was pointed out that a *suitable* test of the material must be made for determining the critical value; this value is frequently referred to as the maximum utilizable strength of the material. It is important to understand how the theories of failure fit into this picture.

For a given general mode of failure, each theory of failure, as stated in Art. 31, names the (significant) quantity which is the cause of failure when the value of the quantity reaches the critical value, and it also states that a tension test is a suitable test for determining the critical or maximum value of this significant quantity.

It is important to note that if an appropriate or suitable test could always be selected so that the material would be subjected to the same conditions of stress that it is subjected to in the actual member, there would be no need for theories of failure. For example, in Table 1 the maximum utilizable strength of a material as determined by each of the several theories of failure is obtained from a tension test, and hence in the design of any member in which the state of stress is uniaxial the member would be given the same dimensions by all the theories of failure. Similarly, if the maximum or limiting values of the various quantities that are considered to be the cause of failure were obtained from a torsion test of the material, any member of the same material subjected to a state of stress of pure shear would be given the same dimensions by all the theories of failure, since all the quantities assumed to cause failure would reach their limiting values simultaneously.

If a theory of failure were correct under all conditions in which load-resisting members are used, it would predict the nature of the quantity (stress, strain, or energy, etc.) and the limiting value thereof (as obtained from a specified test) which would limit the load that could be applied to the member without causing structural damage.

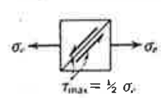
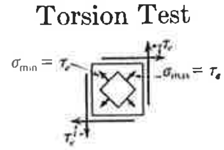
This, however, is too much to expect from a theory of failure when consideration is given to the radical difference in the modes of failure (ranging from incipient yielding to brittle fracture) and to the simplifying conditions that are necessary to impose on a suitable test. In general, we are limited, because of practical considerations, to one of two tests, namely, the tension test or the torsion test.

In interpreting the theories of failure a given general mode of failure is understood to occur. The significance of the theories will here be studied by assuming that failure occurs when inelastic strain (yielding) starts and by comparing the limiting values of the significant quantities stated in the theories, when the limiting values are obtained for each of two states of stress. The first state of stress is a uniaxial stress as exists in the tension test, and the second is a biaxial state of stress corresponding to pure shear as exists in the torsion test.

In Table 2 is shown a comparison of the limiting value (maximum utilizable strength of a material) as obtained, by each of the theories of

TABLE 2

COMPARISON OF MAXIMUM UTILIZABLE STRENGTHS OF A MATERIAL ACCORDING TO VARIOUS THEORIES OF FAILURE FOR EACH OF TWO STATES OF STRESS AS OCCUR IN THE TENSION AND TORSION TESTS

(1) Theory of Failure	(2) Maximum Utilizable Strength as Obtained from a Tensile Test 	(3) Maximum Utilizable Strength as Obtained from a Torsion Test 	(4) Relation between Values of σ_e and τ_e if the Theory of Failure Were Correct for Both States of Stress (col. 2 = col. 3)
Maximum normal stress theory	σ_e	τ_e	$\tau_e = \sigma_e$
Maximum strain theory: $\mu = \frac{1}{4}$	$\frac{\sigma_e}{E}$	$\frac{5}{4}\frac{\tau_e}{E}$	$\tau_e = 0.80\sigma_e$
Maximum shearing stress theory	$\frac{1}{2}\sigma_e$	τ_e	$\tau_e = 0.50\sigma_e$
Maximum octahedral stress theory	$\frac{\sqrt{2}}{3}\sigma_e$	$\frac{\sqrt{2}}{\sqrt{3}}\tau_e$	$\tau_e = 0.577\sigma_e$
Maximum energy of distortion	$\frac{1+\mu}{3}\frac{\sigma_e^2}{E}$	$(1+\mu)\frac{\tau_e^2}{E}$	$\tau_e = 0.577\sigma_e$

failure, from the tensile test (column 2) and from the torsion test (column 3). These values should be equal if all theories are correct. The relationships found by equating the two for each theory are given in column 4.

The results from tests of many ductile metals show that the shearing elastic limit (or yield strength) τ_e found from the torsion test varies from about 0.55 to 0.60 of the tensile yield strength σ_e found from the tension test, an average value being about 0.57.

The results of Table 2, therefore, indicate that the energy of distortion theory or its equivalent, the octahedral shearing stress theory, is the most satisfactory theory of failure of a ductile metal under static load for which the maximum utilizable value of the energy of distortion or of the octahedral shearing stress is found from the tension test. However, the maximum shearing stress theory, under the same conditions of stress, is also a reasonably satisfactory theory. It gives a value of the maximum utilizable strength τ_e of the material which is about 15 per cent less than that given by the energy of distortion theory. Thus, it gives values in design on the safe side. It is widely used for design of ductile metals under conditions of static loading and of ordinary temperatures in which creep is not of importance. It is also clear from Table 2 that the theories of maximum principal stress and the maximum principal strain are applicable only when the maximum principal stress in the material is very large relative to the maximum shearing stress at the same point so that the failure is by fracture rather than by yielding.

The states of stress in the tension and torsion tests represent about as wide a range of stress conditions as occurs in most engineering members that fail by yielding under static loads. In the tension test $\sigma_{\max}/\tau_{\max} = 2$, and in the torsion test $\sigma_{\max}/\tau_{\max} = 1$. For some triaxial states of stress $\sigma_{\max}/\tau_{\max}$ is greater than 2, approaching infinity when the triaxial stresses are equal and of like sign, but failure then becomes one of brittle fracture, if the stresses are tensile stresses.

For states of stress in which $\sigma_{\max}/\tau_{\max}$ lies between 2 and 1 as, for example, in a cylindrical shaft subjected to a bending moment M and a torsional moment T producing the state of stress shown in Figs. 47, 49, and 50, the results given by the various theories are shown in Fig. 47. The diameter d, which is just large enough to prevent inelastic action in the shaft, is computed by each theory of failure, and these values of d are then compared by obtaining the ratios of the various values of d to the value d_s, computed by the maximum shearing stress theory.

These ratios are obtained for combinations of T and M ranging from M acting alone ($T/M = 0$) to T acting alone. (The combination for which $T/M = \infty$, or T acting alone, is shown by the horizontal lines at the right side of the figure which are asymtotes for the curves indicated by the arrows.) It will be noted that the maximum shearing stress theory gives the largest diameter and the maximum normal stress theory the smallest diameter for all ratios of T to M, except for $T/M = 0$, where all diameters are equal.

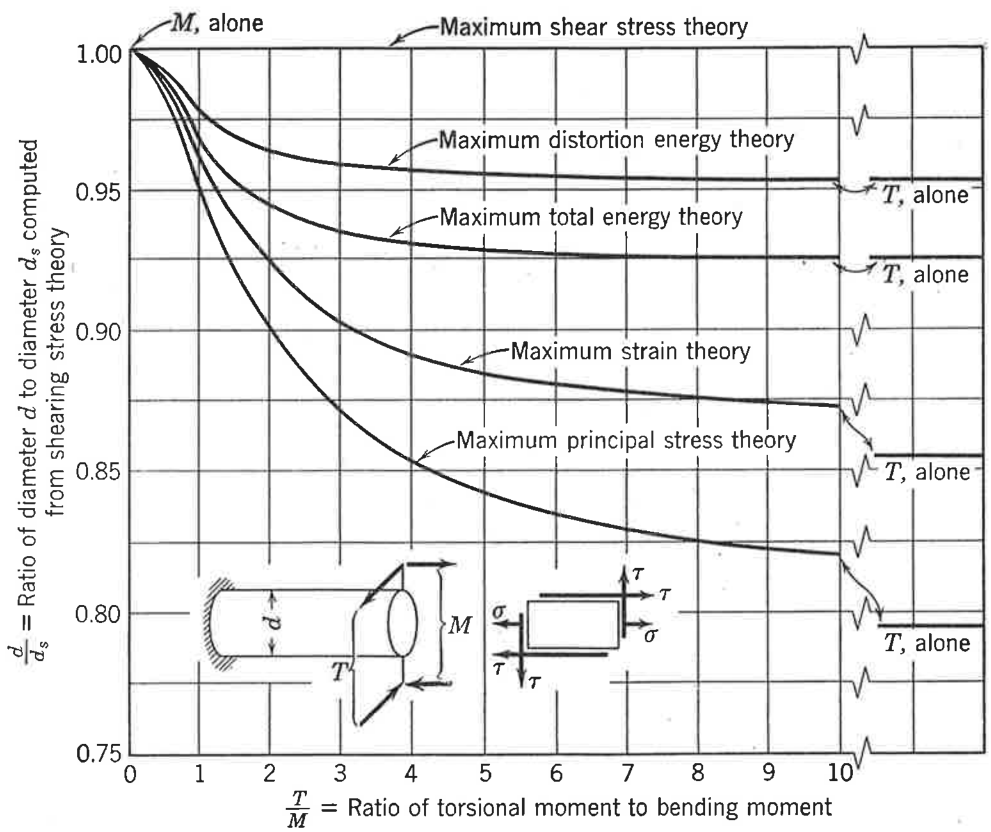


FIG. 47 Comparison of theories of failure.

Figure 48 compares in another way the two most appropriate theories of failure when the mode of failure is by yielding and when the state of stress is the same as that considered in Fig. 47, and it also covers a range in stress from $\sigma_{\max}/\tau_{\max} = 2$ (bending alone) to $\sigma_{\max}/\tau_{\max} = 1$ (torsion alone). The equations represented by these curves are found as follows: For any combination of τ and σ, yielding starts according to the maximum shearing stress theory when

$$\sqrt{(\sigma/2)^2 + \tau^2} = \sigma_e/2 \qquad \text{or} \qquad 4(\tau/\sigma_e)^2 + (\sigma/\sigma_e)^2 = 1 \qquad (70)$$

Likewise, yielding starts according to the energy of distortion theory when

$$\frac{1+\mu}{3E}(\sigma^2 + 3\tau^2) = \frac{1+\mu}{3E}\sigma_e^{\,2} \qquad \text{or} \qquad 3\left(\frac{\tau}{\sigma_e}\right)^2 + \left(\frac{\sigma}{\sigma_e}\right)^2 = 1 \quad (71)$$

Other Factors To Be Considered. The theories of failure, however, do not take account of all the conditions that the engineer must consider in the problem of failure, even of failure of ductile material subjected to static loads at ordinary temperatures. In the theories as here stated it is assumed that the failure occurs when inelastic action starts. In many

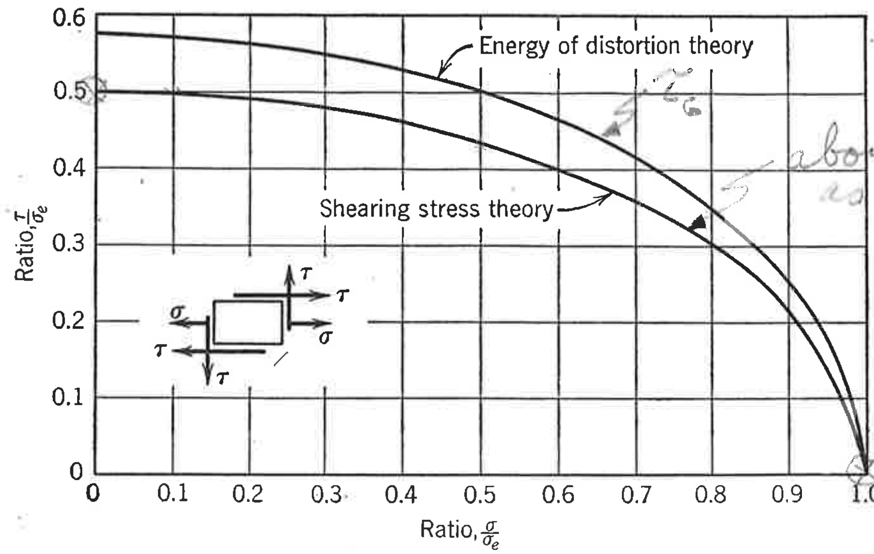


FIG. 48 Comparison of two of the theories of failure.

uses of load-resisting members some inelastic strain may occur without destroying the usefulness of the member, and these inelastic strains cause a readjustment of stresses which may permit an appreciable increase in the loads on the member. This topic is discussed in Part V.

Furthermore, if the theories are applied to a type of failure different from that assumed in the above statement of the theories, a different type of test for determining the limiting values of the quantities considered to be the cause of failure is required. For example, if the failure in a ductile material results from highly localized action in the material, such as failure caused by many (repeated) applications of the load (fatigue failure), the test of the ductile material for determining the limiting value would *not* be a *static* tensile test in which a relatively large amount of material is involved in the failure, but would be a series of repeated load (fatigue) tests in which localized action controls the failure. The value obtained from such a test usually is the endurance limit, and the limiting values required in the application of the various theories of failure would be expressed in terms of this endurance limit. This is done in Chapter 12.

33 Application of theories of failure. *Design formulas.* As stated in the preceding article, the dimensions that should be assigned to a ductile member which is to be subjected to static loads depend on the theory held concerning the cause of the breakdown of elastic action (yielding). This fact is illustrated by Fig. 47 for a cylindrical shaft which is subjected to a bending moment M and a torsional moment T, which produce the state of stress also shown in Fig. 47.

In obtaining design formulas it should be recalled from Art. 2 that the main purpose of the member considered is to resist *loads* safely and that the factor of safety N should be applied in such a way that the design *loads* are increased to N times the loads that cause inelastic action to start.

Maximum Principal Stress Theory. The maximum principal stress σ_{max} must not exceed the tensile elastic strength σ_e (see Table 1). Under these elastic conditions the loads are directly proportional to the stresses, and hence N can be applied to σ_e. Thus the working value of the principal stress is σ_e/N, and the equation for design is

$$\sigma_{max} = \sigma_e/N \tag{72}$$

For the state of stress considered here as shown in Fig. 47, Eq. 72 becomes

$$\sigma_{max} = \tfrac{1}{2}\sigma + \tfrac{1}{2}\sqrt{\sigma^2 + 4\tau^2} = \sigma_e/N \tag{73}$$

in which σ and τ may be expressed in terms of the loads (M and T) acting on the member.

Maximum Shearing Stress Theory. The maximum shearing stress τ_{max} according to Eq. 10 is $\tau_{max} = \frac{1}{2}(\sigma_{max} - \sigma_{min})$. The value of τ_{max} must not exceed $\frac{1}{2}\sigma_e$ (see Table 1). The working value for the shearing stress is, therefore, $\frac{1}{2}(\sigma_e/N)$, and the design equation is

$$\tau_{max} = \tfrac{1}{2}(\sigma_{max} - \sigma_{min}) = \tfrac{1}{2}(\sigma_e/N) \tag{74}$$

For the state of stress shown in Fig. 47, Eq. 74 becomes

$$\tau_{max} = \tfrac{1}{2}\sqrt{\sigma^2 + 4\tau^2} = \tfrac{1}{2}(\sigma_e/N) \tag{75}$$

Maximum Strain Theory. The maximum strain ϵ_{max} is given by Eq. 39

$$\epsilon_{max} = (\sigma_1/E) - \mu(\sigma_2/E) - \mu(\sigma_3/E)$$

where σ_1, σ_2, and σ_3 are the principal stresses, σ_1 having the largest numerical value. Since ϵ_{max} must not exceed $\epsilon_e = \sigma_e/E$ (see Table 1), its working value is ϵ_e/N, and the equation for design is

$$\epsilon_{\max} = (\sigma_1/E) - \mu(\sigma_2/E) - \mu(\sigma_3/E) = (\epsilon_e/N) = (1/E)(\sigma_e/N) \quad (76)$$

In Eq. 76 the working value of strain is equal to $1/E$ times the working value of the stress, and, if the factor $1/E$ which is common to both sides of the equation is cancelled out, the design equation reduces to

$$\sigma_1 - \mu\sigma_2 - \mu\sigma_3 = \sigma_e/N \quad (77)$$

For the state of stress shown in Fig. 47 and for $\mu = \frac{1}{4}$, Eq. 77 becomes

$$\tfrac{3}{8}\sigma + \tfrac{5}{8}\sqrt{\sigma^2 + 4\tau^2} = \sigma_e/N \quad (78)$$

Maximum Total Energy Theory. According to this theory, inelastic action begins when the total energy per unit volume w has the value $w_e = \frac{1}{2}(\sigma_e^2/E)$, which is the total energy absorbed per unit volume at the elastic strength of the material in a standard tension specimen. The foregoing statement as applied to a state of triaxial stress is, from Eq. 57,

$$w = \frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)] = w_e = \frac{1}{2}\frac{\sigma_e^2}{E} \quad (79)$$

If the loads are proportional to the stresses, the energy w, as given by Eq. 79, is a function of the loads to the second power. Hence, the factor of safety N must be applied (see Art. 2) to the quantity $\sqrt{w}$ in order to limit the load to $1/N$ times the loads which will cause inelastic action to begin. Therefore, in Eq. 79 the square root of both sides of the equation is taken, and then the factor of safety N is applied. Thus

$$\sqrt{\frac{1}{2E}[\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)]} = \frac{\sqrt{w_e}}{N} = \frac{\sigma_e}{N}\sqrt{\frac{1}{2E}} \quad (80)$$

If the factor $\sqrt{1/2E}$, which is common to both sides of the equation, is cancelled out, the design equation reduces to

$$\sqrt{\sigma_1^2 + \sigma_2^2 + \sigma_3^2 - 2\mu(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1)} = \sigma_e/N \quad (81)$$

For the state of stress as shown by Fig. 47, Eq. 81 becomes

$$\sqrt{\sigma^2 + 2(1 + \mu)\tau^2} = \sigma_e/N \quad (82)$$

Maximum Energy of Distortion Theory. In this theory inelastic action is assumed to begin when the energy of distortion per unit volume,

w_d, becomes equal to the value $w_{de} = [(1 + \mu)\sigma_e^2]/3E$, which is the energy of distortion per unit volume absorbed at the elastic strength of the material in a standard tensile specimen. This theory of failure is applied to a state of triaxial stress by using the expression for w_d given in Eq. 60. The design equation then may be written

$$w_d = \frac{1 + \mu}{6E}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]$$

$$= w_{de} = \frac{1 + \mu}{3E}\sigma_e^2 \tag{83}$$

The square root of both sides of Eq. 83 is taken, and then the factor of safety is applied, for the same reason given in the foregoing treatment of the total energy. Thus

$$\sqrt{\frac{1 + \mu}{6E}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]} = \frac{\sqrt{w_{de}}}{N}$$

$$= \sqrt{\frac{1 + \mu}{3E}} \cdot \frac{\sigma_e}{N} \tag{84}$$

This equation may be simplified by canceling out the common factor $(1 + \mu)/3E$ leading to the design equation

$$\sqrt{\tfrac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]} = \sigma_e/N \tag{85}$$

For a state of stress as shown by Fig. 47, Eq. 85 reduces to

$$\sqrt{\sigma^2 + 3\tau^2} = \sigma_e/N \tag{86}$$

Maximum Octahedral Shearing Stress Theory. In this theory of failure, inelastic action is assumed to begin when the shearing stress on the octahedral planes (planes making equal angles with the planes on which the three principal stresses act) becomes equal to the value $\tau_{Ge} = \sqrt{2}\sigma_e/3$, which is the value of the octahedral stress occurring at the elastic strength of the material in a standard tensile specimen. When there are three principal stresses, the octahedral shearing stress is given by Eq. 26. Thus the design equation is

$$\tau_{G_{\max}} = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2} = \frac{\sqrt{2}}{3}\frac{\sigma_e}{N} \tag{87}$$

For the state of stress as shown in Fig. 47, Eq. 87 reduces to

$$\tau_{G_{\max}} = \frac{\sqrt{2}}{3}\sqrt{\sigma^2 + 3\tau^2} = \frac{\sqrt{2}}{3}\,\frac{\sigma_e}{N} \tag{88}$$

or simply

$$\sqrt{\sigma^2 + 3\tau^2} = \sigma_e/N \tag{89}$$

which is the same as Eq. 86.

Illustrative Problems

Problem 35. A cylindrical shaft made of steel for which the tensile elastic strength (yield strength) is $\sigma_e = 100{,}000$ lb per sq in. is subjected to static loads consisting of a bending moment $M = 100{,}000$ lb-in. and a torsional moment $T = 300{,}000$ lb-in. as indicated in Fig. 47. Assume that for steel $E = 30 \times 10^6$, $\mu = 0.25$. Determine the diameter d which the shaft must have for a factor of safety of 2.

Solution. The four steps in the rational procedure in design as outlined in Art. 2 (Chapter 1) will guide the solution. The failure results from yielding; therefore, in accordance with the discussion in Art. 32, the energy of distortion theory of failure, or its equivalent, the octahedral shearing stress theory, should give the most satisfactory results. The use of the shearing stress theory also can be justified, as giving conservative values. Both theories will be used.

MAXIMUM SHEARING STRESS THEORY. The design equation according to this theory of failure is (see Eq. 75)

$$\tau_{\max} = \tfrac{1}{2}\sqrt{\sigma^2 + 4\tau^2} = \tau_e/N = \tfrac{1}{2}(\sigma_e/N)$$

in which $\sigma = Mc/I = 32M/\pi d^3$, and $\tau = Tc/J = 16T/\pi d^3$. Thus

$$(16/\pi d^3)\sqrt{M^2 + T^2} = \tfrac{1}{2}(\sigma_e/N)$$

$$(1{,}600{,}000/\pi d^3)\sqrt{(1)^2 + (3)^2} = \tfrac{1}{2}(100{,}000/2) = 25{,}000$$

Hence $$d = 4.01 \text{ in.}$$

OCTAHEDRAL SHEARING STRESS THEORY. The design equation according to this theory is (see Eq. 89 or the same equation as obtained by the energy of distortion theory, Eq. 86)

$$(16/\pi d^3)\sqrt{4M^2 + 3T^2} = \sigma_e/N$$

$$(1{,}600{,}000/\pi d^3)\sqrt{4(1)^2 + 3(3)^2} = 50{,}000$$

Hence $$d = 3.83 \text{ in.}$$

Thus a diameter not less than 3.83 in. would be justified for strength.

Problem 36. A cylindrical bar of cast iron is subjected to a bending moment of $M = 10{,}000$ lb-in. and a torsional moment of $T = 30{,}000$ lb-in., as shown in Fig. 47. Assume that for cast iron $\sigma_e = 30{,}000$ lb per sq in., $E = 15 \times 10^6$, and $\mu = 0.25$.

Determine the minimum diameter the bar should have, based on a factor of safety of 3.

Solution. The four steps in a rational design procedure as outlined in Art. 2 (Chapter 1) are involved in the solution. The bar will fail by brittle fracture. Therefore, in accordance with the discussion in Art. 31, the maximum principal stress theory and the maximum principal strain theory are the most satisfactory theories. Both theories will be used in the solution.

MAXIMUM PRINCIPAL STRESS THEORY. The design formula (Eq. 73) is

$$\sigma_{\max} = \tfrac{1}{2}\sigma + \tfrac{1}{2}\sqrt{\sigma^2 + 4\tau^2} = \sigma_e/N$$

in which $\sigma = Mc/I = 32M/\pi d^3$ and $\tau = Tc/J = 16T/\pi d^3$

Thus $\tfrac{1}{2}(32M/\pi d^3) + \tfrac{1}{2}\sqrt{(32M/\pi d^3)^2 + 4(16T/\pi d^3)^2} = \sigma_e/3$

$$\tfrac{1}{2}(32/\pi d^3)(M + \sqrt{M^2 + T^2}\,) = 10{,}000$$

$$(160{,}000/\pi d^3)[1 + \sqrt{(1)^2 + (3)^2}\,] = 10{,}000$$

$$\pi d^3 = 16 \times 4.16 = 66.56$$

Hence $d = 2.77$ in.

MAXIMUM PRINCIPAL STRAIN THEORY. According to this theory of failure the design formula is (see Eq. 78)

$$\epsilon_{\max} = (\sigma_1/E) - \mu(\sigma_2/E) = \epsilon_e$$

or

$$\tfrac{3}{8}\sigma + \tfrac{5}{8}\sqrt{\sigma^2 + 4\tau^2} = \sigma_e/N$$

Hence

$$(32/\pi d^3)(\tfrac{3}{8}M + \tfrac{5}{8}\sqrt{M^2 + T^2}\,) = 30{,}000/3$$

$$(320{,}000/\pi d^3)[\tfrac{3}{8} \times 1 + \tfrac{5}{8}\sqrt{(1)^2 + (3)^2}\,] = 10{,}000$$

$$\pi d^3 = 75.2$$

Hence $d = 2.88$ in.

Thus, if fracture occurs because the strain reaches a limiting value, the shaft should have a diameter of 2.88 in. in order to prevent the shaft from fracturing when the loads on the shaft are three times the actual loads. The maximum strain theory seems to fit the small amount of test data that are available somewhat better than does the maximum principal stress theory.

Problems

37. A pressure P of 10,000 lb on the crank pin of the crank shaft in Fig. 49 is required to turn the shaft at constant speed. The crank shaft is made of ductile steel having a tensile (and compressive) elastic limit or yield strength of 40,000 lb per sq in. as found from a tension (or compression) test. Assume that $E = 30 \times 10^6$ and $\mu = 0.25$. Calculate the diameter of the shaft based on a factor of safety of 2. In

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