Markov Chains

Several of the most powerful analytic techniques for evaluation computer system performance (and many other systems) are based on the theory of Markov chains. A Markov chain is a special case of a Markov process, which is itself a special case of a random process.

Random (stochastic) process $\equiv$ family of (ordered set of related) random variables $X(t)$ where t is an indexing parameter (usually time)

There are many kinds of random processes. Two of the most important distinguishing characteristics of a random process are whether or not the values that the random process can take on are continuous over some interval(s) and whether or not the indexing parameter is continuous or discrete.

Classification of random processes:

(1) state space

continuous-state - $X(t)$ can take on any value over a finite or infinite continuous interval or set of such intervals

discrete-state - $X(t)$ has only a finite or countable number of possible values $\{s_0, s_1, s_2, \ldots, s_i, \ldots\}$
- usually referred to as a chain

(2) index parameter (call it time)

discrete-time - permitted times at which changes in value may occur are finite or countable ($X(t)$ may be represented as a set $\{X_i\}$)

continuous-time - changes may occur anywhere within a finite or infinite interval on the time axis or set of such intervals
- often called a random process

The state of a continuous-time chain at a time t is $X(t)$; the state of a discrete-time chain at time n is X_n.

Markov chains

A Markov chain is a discrete-state random process in which the only state that influences the next state is the current state. To be more precise:

Discrete-time Markov chain:

X_{n+1} depends only on X_n and not on any X_i, $1 \le i < n$

$$\Pr\left[X_{n+1} = s_i \middle| X_n = s_j, X_{n-1} = s_k, \ldots, X_1 = s_l\right]$$
$$= \Pr\left[X_{n+1} = s_i \middle| X_n = s_j\right]$$

This equation is referred to as the Markov property.

Continuous-time Markov chain:

Consider a continuous-time random process in which the number of times the random variables $X(t)$ change value (the process changes state) is finite or countable. Let $t_1, t_2, t_3, \ldots$ be the times at which the process changes state. If we ignore how long the random process remains in a given state, we can view the sequence $\left\{X_{t_1}, X_{t_2}, X_{t_3}, \ldots\right\}$ as a discrete-time process embedded in the continuous-time process.

A continuous-time Markov chain is a continuous-time, discrete-state random process such that

(1) the embedded discrete-time process is a discrete-time Markov chain, and

(2) the time between state changes is a random variable with a memory-less distribution.

A distribution function $F_T(\cdot)$ is memory-less if and only if

$$F_T(t) = F_T(t + \tau | T > \tau)$$

This says that the distribution of the time until the next state change is not a function of the time since the last state change.

We can restate this as

$$F_T(t) = \Pr\left[T \le t + \tau | T > \tau\right]$$

Using the definition of conditional probability,

$$F_T(t) = \frac{\Pr\left[T \le t + \tau \,\&\, T > \tau\right]}{\Pr\left[|T > \tau\right]}$$
$$= \frac{F_T(t + \tau) - F_T(\tau)}{1 - F_T(\tau)}$$

Dividing both sides by t and taking the limit as $t \to 0$,

$$\lim_{t\to 0}\frac{F_T(t)}{t} = \lim_{t\to 0}\frac{F_T(t+\tau)-F_T(\tau)}{t\left(1-F_T(\tau)\right)}$$

$$F_T^{\copyright}(0) = \frac{F_T^{\copyright}(\tau)}{1-F_T(\tau)}$$

$$F_T^{\copyright}(\tau) + F_T^{\copyright}(0)F_T(\tau) - F_T^{\copyright}(0) = 0$$

The solution to this linear, first-order differential equation is

$$F_T(t) = 1 - e^{-F_T^{\copyright}(0)t}$$

Hence, the only continuous-time, memory-less distribution is the exponential distribution, and the time between state changes in a continuous-time Markov chain is exponentially distributed.

For discrete-time Markov chains, the next state may be the same as the current state: $X_{n+1} = X_n$. Let $p = \Pr\left[X_{n+1} = s_i \mid X_n = s_j\right]$ for all $n \geq 0$ (this will be true for all homogeneous Markov chains). The probability that X_{n+1} is different than X_n is $1-p$. The probability that X_{n+1} is the same as X_n and X_{n+2} is different is $p(1-p)$. In general,

$$\Pr\left[X_{n+i} \neq s_i \ \& \ X_{n+i-1} = X_{n+i-2} = \ldots = X_{n+1} = s_i \mid X_n = x_n\right] = p^{i-1}(1-p)$$

Therefore, the number of state transitions between state changes is geometrically distributed.

One special type of Markov chain is a birth and death process, in which the states take on all non-negative integer values on a (possibly infinite) range; that is, $\{s_0, s_1, s_2, \ldots, s_i, \ldots\} = \{0,1,2,\ldots,i,\ldots\}$. In this case, we can just refer to s_i as i, and define a birth and death process as:

If $X_n = i$, then $X_{n+1} = i+1$, i, or $i-1$

i.e., state transitions are always between neighboring states.