

Comp 360 -- Midterm Exam

The following are the rules for this examination:

- a three hour time limit to finish the entire exam;
- you need not do the entire exam in one session, but once you read a problem you must finish the problem without taking time out (except for obvious emergencies);
- Please write all your answers in INK, not in pencil! Pencil is difficult to read. If you want to change an answer, just cross out your old answer and write your new answer. Do not try to erase.
- closed textbook, closed notes; some helpful identities appear on pages 2,3.
- all work must be entirely your own;
- you may not confer on any problems with any person except the professor;
- the exam must be handed in to my office no later than Noon on Wednesday, October 25.

There are 5 problems on the exam. Each problem is worth 50 points. For full credit you must do all 5 problems.

Good luck.

Dot Product

$$u \cdot v = |u| |v| \cos(\theta) \quad (\text{definition})$$

$$u \cdot v = v \cdot u \quad (\text{commutative})$$

$$u \cdot (v + w) = u \cdot v + u \cdot w \quad (\text{distributive})$$

$$(v + w) \cdot u = v \cdot u + w \cdot u$$

$$|v|^2 = v \cdot v \quad (\text{length})$$

$$\cos(\theta) = \frac{u \cdot v}{|u| |v|} \quad (\text{angle})$$

$$u_{\parallel} = \left(\frac{u \cdot v}{v \cdot v} \right) v \quad (\text{parallel projection})$$

$$u_{\parallel} = (u \cdot v)v \quad \text{if } |v| = 1$$

$$u_{\perp} = u - u_{\parallel} = u - \left(\frac{u \cdot v}{v \cdot v} \right) v \quad (\text{perpendicular projection})$$

$$u_{\perp} = u - (u \cdot v)v \quad \text{if } |v| = 1$$

$$u \cdot v = 0 \Leftrightarrow u \perp v \quad (\text{orthogonality})$$

Cross Product

$$|u \times v| = |u| |v| \sin(\theta) \quad (\text{definition})$$

$$u \times v \perp u, v$$

$$\text{sgn}(u, v, u \times v) > 0$$

$$\text{area}(u, v) = |u \times v| \quad (\text{area})$$

$$(u \times v) \cdot u = 0 \quad (\text{orthogonality})$$

$$(u \times v) \cdot v = 0$$

$$u \times u = 0 \quad (\text{parallelism})$$

$$u \times v = 0 \Leftrightarrow v \parallel \pm u$$

$$u \times (v + w) = u \times v + u \times w \quad (\text{distributive})$$

$$(v + w) \times u = v \times u + w \times u$$

$$u \times v = -v \times u \quad (\text{anti-commutative})$$

$$u \times (v \times w) = (u \cdot w)v - (u \cdot v)w \quad (\text{non-associative})$$

$$(v \times w) \times u = (u \cdot v)w - (u \cdot w)v$$

$$|u \times v|^2 = |u|^2 |v|^2 - (u \cdot v)^2 \quad (\text{length})$$

$$(u_1 \times u_2) \cdot (v_1 \times v_2) = (u_1 \cdot v_1)(u_2 \cdot v_2) - (u_1 \cdot v_2)(u_2 \cdot v_1) \quad (\text{Lagrange Identity})$$

Determinant

$$\det(u, v, w) = (u \times v) \cdot w$$

(definition)

$$\text{vol}(u, v, w) = |\det(u, v, w)|$$

(volume)

$$\det(u, v, w) > 0 \Leftrightarrow \text{sgn}(u, v, w) > 0$$

(orientation)

$$\det(u, v, w) \neq 0 \Leftrightarrow u, v, w \text{ are linearly independent}$$

(linear independence)

$$\det(u, u, w) = \det(u, v, u) = \det(u, v, v) = 0$$

$$\det(u, v, w) = \det(v, w, u) = \det(w, u, v)$$

(skew symmetry)

$$\det(v, u, w) = -\det(u, v, w)$$

$$\det(u_1 + c u_2, v, w) = \det(u_1, v, w) + c \det(u_2, v, w)$$

$$\det(u, v_1 + c v_2, w) = \det(u, v_1, w) + c \det(u, v_2, w)$$

(multi-linearity)

$$\det(u, v, w_1 + c w_2) = \det(u, v, w_1) + c \det(u, v, w_2)$$

1. (50 points)

Consider the pentagonal bump and the corresponding bump fractal depicted in Figure 1:

- a. Write a turtle program to generate the pentagonal bump.
- b. Write a turtle program to create the corresponding bump fractal.
- c. How many transformations would you need to generate this bump fractal using an iterated function system (IFS)?
- d. Describe four of the transformations in the iterated function system (IFS) that generate this fractal curve.
- e. Compute the fractal dimension of this fractal curve.

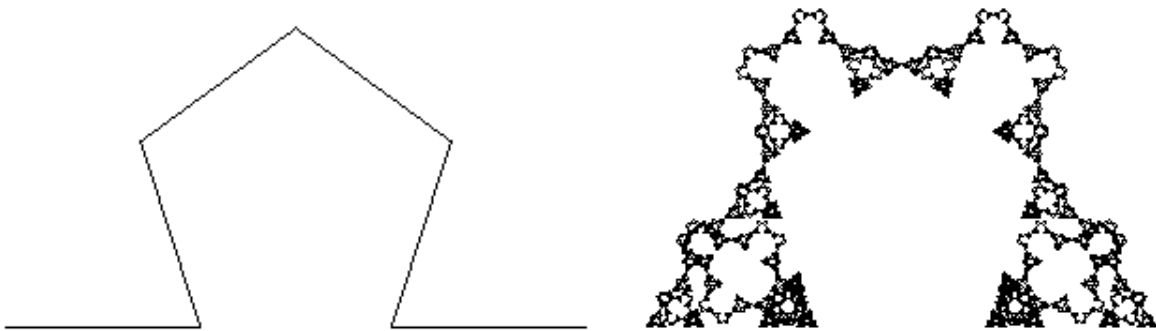


Figure 1: A pentagonal bump and the corresponding bump fractal.

2. (50 points)

Consider the fractal tree depicted in Figure 2:

- a. Write a turtle program to generate this fractal tree.
- b. Describe the transformations in the iterated function system (IFS) that generate this fractal tree.
- c. Describe the condensation set of this fractal tree.

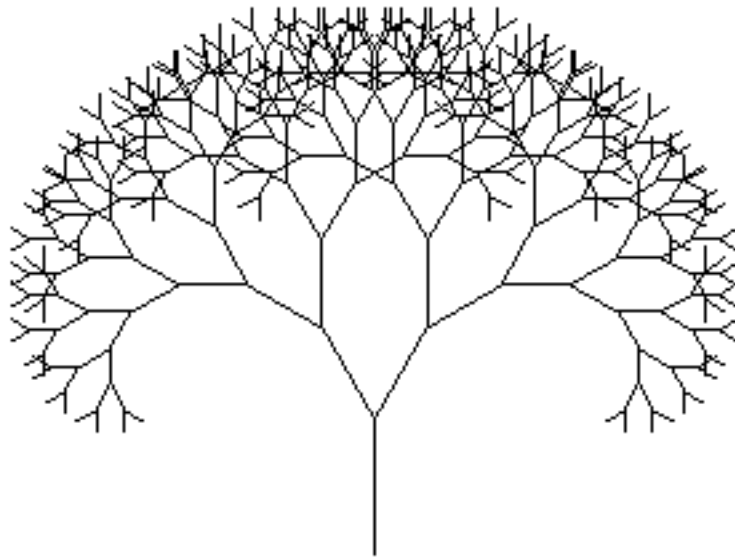


Figure 2: A fractal tree.

3. (50 points)

- a. Show that in 2-dimensions an affine transformation is uniquely determined by the image of two points and one vector that is not parallel to the line determined by the two points.
- b. Let P_1^*, P_2^*, v^* be the images of P_1, P_2, v under the affine transformation A . Describe the 3×3 matrix M that represents the affine transformation A .

4. (50 points)

Without appealing to coordinates, prove that:

a. $|u \times v|^2 = |u|^2 |v|^2 - (u \cdot v)^2$

b. $u \times (v \times w) + v \times (w \times u) + w \times (u \times v) = 0.$

c. $(u_1 \times u_2) \times (v_1 \times v_2) = \det(u_1, u_2, v_2)v_1 - \det(u_1, u_2, v_1)v_2.$

d. $(u_1 \times u_2) \times (v_1 \times v_2) = \det(u_1, v_1, v_2)u_2 - \det(u_2, v_1, v_2)u_1.$

5. (50 points)

Parallel projection is projection onto a plane S along the direction of a unit vector u . As usual the plane S is defined by a point Q and a unit normal vector n (see Figure 3).

a. Show that under parallel projection for any vector v and any point P

i.
$$v^{new} = v - \frac{v \cdot n}{u \cdot n} u$$

ii.
$$P^{new} = P - \frac{(P - Q) \cdot n}{u \cdot n} u$$

b. Show that the 4×4 matrix representing parallel projection is given by

$$PProj(Q, n, u) = \begin{pmatrix} I - \frac{n^T * u}{u \cdot n} & 0 \\ \frac{Q \cdot n}{u \cdot n} u & 1 \end{pmatrix}.$$

c. Suppose that

- S is the xy -plane,
- u is the unit vector parallel to the vector $(1,1,1)$,

Write down the explicit entries of the 4×4 matrix $PProj(Q, n, u)$.

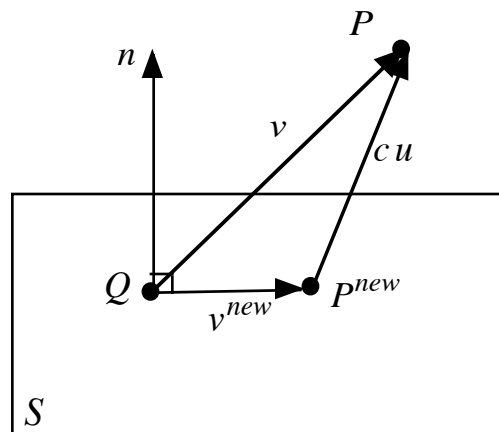


Figure 3: Parallel projection along a direction u into a plane S .